

# Operational Theory of the Eight-Port Homodyne Detection

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## Abstract

The eight-port homodyne detection apparatus is analyzed in the framework of the operational theory of quantum measurement. For an arbitrary quantum noise leaking through the unused port of the beam splitter, the positive operator valued measure and the corresponding operational homodyne observables are derived. It is shown that such an eight-port homodyne device can be used to construct the operational quantum trigonometry of an optical field. The quantum trigonometry and the corresponding phase space Wigner functions are derived for a signal field probed by a classical local oscillator and a squeezed vacuum in the unused port.

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## I. INTRODUCTION

In this paper we present a general operational description of measurements performed with the help of the eight-port homodyne detector. This experimental setup has been extensively used in the past years in connection with the homodyne detection of optical signals [1] and the determination of the phase of quantum states of light [2]. The series of beautiful experiments, performed by Noh, Fougères and Mandel (NFM) has provided a fully operational approach to the measurement of the quantum phase of a single mode light field. These experiments have inspired a lot of further theoretical works devoted to a general operational approach to quantum measurements in quantum optics.

The NFM experiments have been treated theoretically in many different ways [3–6], however our approach will be entirely based on the concept of the operational phase space *propensity* [7] and on the associated with it operational quantum observables [8]. Such a formulation, which is one of many possible descriptions of realistic measurements, is closely connected with the concept of the positive operator valued measure (POVM). In the operational formulation the POVM replaces the spectral measure of the intrinsic quantum observable [9,10].

The operational formalism has been applied to the NFM experimental device in the context of the operational definition of the Hermitian phase operator [8]. In the framework of such an approach, the so called quantum trigonometries for the optical fields have been constructed [11–13]. Recently the operational theory has been applied to the homodyne detection scheme with a fixed or a random phase of the local classical oscillator [14].

The advantage of using the operational measurement theory is that we can easily and naturally take into account the influence of the experimental device on the measured system. For example, the eight port homodyne detector is characterized by the efficiency of the detectors, the reflectance and the transmittance of the beam splitters, and the state of the quantum field entering the unused ports of the device. This additional “noise” field plays a fundamental role, if the NFM device is used to investigate the phase of quantum optical fields. In such a setup the impact of the correlations on the probe field and the noise field plays a crucial role in the operational definition of the quantum phase. This particular interplay between the quantum signal and the quantum noise, leaking through the unused port in the NFM device, provides the framework of the operational approach to the quantum phase. The case of the noise field being the vacuum state has been analyzed, in the spirit of the operational formalism, in Ref. [11]. These results have been generalized for the case of the squeezed vacuum state [12]. In this paper we give a more detailed discussion of the squeezed vacuum state case and present general results obtained for an arbitrary noise field.

In Section II we shortly review the operational approach to the measurement in quantum mechanics. In Section III we derive the operational probability density — propensity for an arbitrary noise field in the NFM apparatus. In Section IV we derive the operational homodyne observables for the eight-port homodyne detection scheme. Section V is devoted to the discussion of the quantum trigonometry, generated in the operational phase measurements in the NFM device, with the noise being a squeezed vacuum. Finally, some concluding remarks are presented.

## II. OPERATIONAL APPROACH TO MEASUREMENTS IN QUANTUM MECHANICS

The approach to the quantum mechanical theory of measurement that will be used in this paper was described in details in Ref. [8]. Below we just recall the main results of this work in order to make the paper self-contained. The statistical outcomes of an ideal measurement of a certain observable  $\hat{A}$ :  $\hat{A}|a\rangle = a|a\rangle$  are described by the spectral measure [15]:

$$p_\psi(a) = |\langle a|\psi\rangle|^2, \quad (1)$$

where  $|\psi\rangle \in \mathcal{H}$  is the state vector of the measured system. It is known that the spectral measure contains all the relevant statistical information about the investigated system but it makes no reference to the apparatus employed in the actual measurement. Due to this property  $\hat{A}$  will be called an *intrinsic quantum observable*.

The description of more realistic measurements necessarily involves additional degrees of freedom such as, for example, the unused ports of the beam splitters or the local oscillator being the phase calibrating device. In such an approach, in order to describe the quantum mechanical measurement, we have to work in the *extended* Hilbert space  $\mathcal{H} \otimes \mathcal{H}_{\mathcal{F}}$ , containing both the Hilbert space of the investigated system  $\mathcal{H}$  and the Hilbert space  $\mathcal{H}_{\mathcal{F}}$  of all the additional measuring devices, which will be called for short a *filter*. The process of measurement is described by the interaction between the system and the filter. Because we are interested only in the properties of the system, we shall reduce the degrees of freedom by tracing the dynamical evolution in the combined Hilbert space over the filter Hilbert space obtaining in such a way an operational *propensity*  $\text{Pr}(a)$ , which is nothing else but a probability distribution of a certain classical quantity  $a$  measured in a real experiment. This propensity, for arbitrary density operator  $\hat{\rho}$  of the system, is

$$\text{Pr}(a) = \text{Tr}\{\hat{\mathcal{F}}(a)\hat{\rho}\}, \quad (2)$$

with the normalization condition

$$\int da \text{Pr}(a) = 1 \iff \int da \hat{\mathcal{F}}(a) = \hat{1}, \quad (3)$$

where  $\hat{\mathcal{F}}(a)$  is a filter dependent POVM. We see that in a realistic measurement, involving a filter, the spectral decomposition of  $\hat{A}$  is effectively replaced by a positive valued measure  $da \hat{\mathcal{F}}(a)$  [9].

In view of the linear relation between the propensity and the POVM, the operational statistical moments of the measured quantity

$$\overline{a^n} = \int da a^n \text{Pr}(a), \quad (4)$$

define uniquely an algebra of *operational operators*

$$\overline{a^n} \equiv \langle \hat{A}_{\mathcal{F}}^{(n)} \rangle, \quad (5)$$

where

$$\hat{A}_{\mathcal{F}}^{(n)} = \int da a^n \hat{\mathcal{F}}(a). \quad (6)$$

These operators are called operational operators, because they represent quantities measured in a real experiment i.e., described by a dynamical coupling of the measured system and the filter. As a rule, the algebraic properties of the set  $\hat{A}_{\mathcal{F}}^{(n)}$  differ significantly from those of the powers of  $\hat{A}$ . For instance,  $(\hat{A}_{\mathcal{F}}^{(1)})^2 \neq \hat{A}_{\mathcal{F}}^{(2)}$ . This property will have important consequences in the discussion of the uncertainty introduced by the measurement.

It is seen that the propensity and the operational operators are natural equivalents of the spectral probability distribution and the intrinsic operators. The difference being that the operational observables carry information about the system under investigation and the selected measuring device, which is represented in this paper by the eight-port homodyne detector.

### III. NFM APPARATUS WITH AN ARBITRARY NOISE FIELD

In this Section we derive a general formula for the POVM that describe the NFM device with an arbitrary noise field leaking through the unused port of the beam splitter. We shall assume that the NFM apparatus has unit detectors efficiency, that a 50%:50% beam splitters are used and that the local oscillator is classical. We assume that we want to measure an observable  $\hat{A}(\hat{b}, \hat{b}^\dagger)$ . In general such an observable requires additional informations about the particular ordering of the boson creation  $\hat{b}^\dagger$  and annihilation  $\hat{b}$  operators. For example the quantum observable that corresponds to the polar decomposition of the amplitude of a field mode could be defined as  $\hat{b}/\sqrt{\hat{b}^\dagger \hat{b}}$ , however this formula is not unique because of the possible different ordering of the boson operators. Due to the ordering problem, there is no canonical way of selecting the right, i.e., the physically justified ordering. The operational approach removes this ordering ambiguity. We shall perform a measurement of this observable using the NMF device. In this case the filter Hilbert space is reduced to a Hilbert space  $\mathcal{H}_v$ , where  $v$  denotes the quantized mode leaking through the unused port. In the extended filter-system space  $\mathcal{H}_b \otimes \mathcal{H}_v$  the observable  $\hat{A}$  will be the following function of the creation and annihilation operators

$$\hat{A} = \hat{A}(\hat{b} + \hat{v}^\dagger, \hat{b}^\dagger + \hat{v}), \quad (7)$$

where the operators  $\hat{b}, \hat{b}^\dagger$  represent the probe field and the operators  $\hat{v}, \hat{v}^\dagger$  describe the noise field, i.e. the filter degrees of freedom. In this extended space, the combination  $\hat{b} + \hat{v}^\dagger$  do commute with  $\hat{b}^\dagger + \hat{v}$  and such operators can be measured simultaneously. In the case of the unimodular phase operator the combination

$$\frac{\hat{b} + \hat{v}^\dagger}{\sqrt{(\hat{b}^\dagger + \hat{v})(\hat{b} + \hat{v}^\dagger)}} \quad (8)$$

is operationally unique and the extended field modes can be simultaneously measurable. A similar approach to the measurement of a simple quantum system, using an extension by an additional degree of freedom, has been used by Einstein Podolsky and Rosen [16]. In their case the particle system with the non commuting position and momentum operators  $\hat{p}$  and

$\hat{q}$  has been extended to commuting observables  $\hat{p} + \hat{P}$  and  $\hat{q} - \hat{Q}$ , where  $\hat{P}$  and  $\hat{Q}$  can be seen as the momentum and the position of the filter. This physical procedure has a counterpart in mathematics, and is called the Naimark extension of the POVM into a projective measure on a larger Hilbert space [10].

The set of operational operators corresponding to the powers of the operator  $\hat{A}$  in the extended filter-system are defined as

$$\hat{A}_{\mathcal{F}}^{(n)} \equiv \text{Tr}_v \{ \hat{A}^n (\hat{b} + \hat{v}^\dagger, \hat{b}^\dagger + \hat{v}) \hat{\varrho}(\hat{v}^\dagger, \hat{v}) \}. \quad (9)$$

We will assume that the noise field density matrix  $\hat{\varrho}(\hat{v}^\dagger, \hat{v})$  can be expressed in the normally order form i.e., all the annihilation operators are to the right of the creation operators. This is true for all physically interesting cases discussed in the literature. Provided that this condition is fulfilled we can rewrite the above expression, using the coherent state decomposition of unity, in the following way

$$\hat{A}_{\mathcal{F}}^{(n)} = \int \frac{d^2\alpha}{\pi} A^n(\alpha, \alpha^*) \hat{\varrho}(\alpha^* - \hat{b}^\dagger, \alpha - \hat{b}). \quad (10)$$

This formula gives us the form of the POVM for the experimental NFM setup with arbitrary noise:

$$\hat{\mathcal{F}}(\alpha) = \hat{\varrho}(\alpha^* - \hat{b}^\dagger, \alpha - \hat{b}), \quad (11)$$

with the normalization:

$$\int \frac{d^2\alpha}{\pi} \hat{\mathcal{F}}(\alpha) = 1. \quad (12)$$

Such a result has a simple meaning in terms of shifts in the coherent state phase space of the signal field. In order to measure the signal mode we have to probe it with a filter state shifted in the phase space by the amount  $\alpha$ , and sum up over all possible manifolds of shifts [7].

As an example of this general approach we find explicitly the POVM for several interesting cases of the filter state. For the vacuum state

$$\hat{\varrho}(\hat{v}^\dagger, \hat{v}) = |0\rangle\langle 0| = : \exp(-\hat{v}^\dagger \hat{v}) : \quad (13)$$

the corresponding POVM is

$$\hat{\mathcal{F}} = |\alpha\rangle\langle\alpha|, \quad (14)$$

where we employed the following identity [17]

$$|\alpha\rangle\langle\alpha| = : \exp[-(\hat{b}^\dagger - \alpha^*)(\hat{b} - \alpha)] : , \quad (15)$$

and have denoted the normal ordering by semicolons. This result has been obtained already in Ref. [11]. Another interesting state is the squeezed vacuum state [18]

$$\begin{aligned} \hat{\varrho}(\hat{v}^\dagger, \hat{v}) &= |s\rangle\langle s| = \cosh^{-1} s \\ &\times : \exp \left[ - \left( \hat{v}^\dagger \hat{v} + \frac{1}{2} \tanh s \left( e^{i\phi} \hat{v}^{\dagger 2} + e^{-i\phi} \hat{v}^2 \right) \right) \right] : \end{aligned} \quad (16)$$

which leads to the POVM in the form

$$\hat{\mathcal{F}} = |\alpha, s\rangle\langle\alpha, s|, \quad (17)$$

which has been derived in Ref. [12]. The squeezed state is generated from the ground state by the action of the squeezing operator  $\hat{S}(s, \phi)$  and the Glauber displacement operator  $\hat{D}(\alpha)$ . Similarly, for the Fock state  $|n\rangle\langle n|$  we obtain

$$\hat{\mathcal{F}} = \hat{D}(\alpha) \frac{1}{\bar{n} + 1} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^{\hat{a}^\dagger \hat{a}} \hat{D}^\dagger(\alpha), \quad (18)$$

which equals to the POVM from the Ref. [8] introduced in the context of the operational position and momentum measurement.

#### IV. OPERATIONAL HOMODYNE OBSERVABLES AND THE POSITION AND MOMENTUM MEASUREMENT

As a first application of the above described formalism we shall discuss the operational measurement of the homodyne quadratures. Such a discussion has already been carried in [14], for a homodyne detection in the standard configuration i.e., with one beam splitter and two detectors. Here we present the operational observables associated with the joint measurements of two field quadratures with the help of the NFM apparatus. We will also slightly generalize our previous considerations, assuming that the beam splitter, which mixes the probe field with the noise field has an arbitrary transmittance coefficient  $T$  and that the noise field is a squeezed vacuum. As before, we shall assume that the local oscillator is a strong coherent field described by  $\alpha = |\alpha| \exp(i\theta)$ .

In the extended filter–system space, the operators leading to the operational observables of the two homodyne quadratures are the “unnormalised” sine and cosine operators [2,6]

$$\hat{X}_1 = \frac{1}{2} \left[ \alpha(\sqrt{T}\hat{b}^\dagger + \sqrt{1-T}\hat{v}) + \alpha^*(\sqrt{T}\hat{b} + \sqrt{1-T}\hat{v}^\dagger) \right] \quad (19)$$

and

$$\hat{X}_2 = \frac{1}{2} \left[ i\alpha(\sqrt{1-T}\hat{b}^\dagger + \sqrt{T}\hat{v}) - i\alpha^*(\sqrt{1-T}\hat{b} + \sqrt{T}\hat{v}^\dagger) \right]. \quad (20)$$

If we define two generating operators ( $i = 1, 2$ ) for the corresponding quadratures

$$\hat{Z}_{i\mathcal{F}}(\xi) \equiv \text{Tr}_v \left\{ \exp(\xi \hat{X}_i) |s_v\rangle\langle s_v| \right\}, \quad (21)$$

then the operational moments are obtained by the  $n$ -fold derivatives:

$$\hat{X}_{i\mathcal{F}}^{(n)} = \left[ \frac{d^n \hat{Z}_{i\mathcal{F}}(\xi)}{d\xi^n} \right]_{\xi=0}. \quad (22)$$

For both quadratures the generating operators can be easily found

$$\hat{Z}_{1\mathcal{F}}(\xi) = \exp \left[ \frac{\xi\sqrt{T}}{2} (\alpha\hat{b}^\dagger + \alpha^*\hat{b}) \right] \exp \left[ \frac{\xi^2}{8}(1-T)|\alpha|^2\sigma_-(s) \right], \quad (23)$$

$$\hat{Z}_{2\mathcal{F}}(\xi) = \exp \left[ \frac{\xi\sqrt{1-T}}{2} (i\alpha\hat{b}^\dagger - i\alpha^*\hat{b}) \right] \exp \left[ \frac{\xi^2}{8}(T)|\alpha|^2\sigma_+(s) \right], \quad (24)$$

where we defined two auxiliary functions

$$\sigma_\pm(s) = \cosh 2s \pm \cos(2\theta - \phi) \sinh 2s \quad (25)$$

and denoted by  $\phi$  the phase of the squeezed vacuum. In the first factor of the generating operators we recognize the intrinsic quadrature operator

$$\hat{x}_\theta = \frac{e^{i\theta}\hat{b}^\dagger + e^{-i\theta}\hat{b}}{\sqrt{2}}, \quad (26)$$

with the rotated quadrature  $\hat{x}_{\theta+\pi/2}$  in  $\hat{Z}_{2\mathcal{F}}(\xi)$ . The second factor of the generating operators may be seen as an operator ordering parameter introduced by Cahill and Glauber [19]. By varying the filter parameters we can select a particular ordering, for example for

$$T = \frac{\sigma_\pm(s)}{1 + \sigma_\pm(s)} \quad (27)$$

the creation and the annihilation operators are ordered antinormally

$$\hat{Z}_{1\mathcal{F}}(\xi) = \exp((\xi/2)\alpha^*\sqrt{T}\hat{a}) \exp((\xi/2)\alpha\sqrt{T}\hat{a}^\dagger). \quad (28)$$

For the discussed experimental NFM scheme, no selection of the parameters can lead to a normal ordering.

The operational quadratures might be calculated explicitly

$$\hat{X}_{1\mathcal{F}}^{(n)} = \left( \frac{\sqrt{1-T}}{2i} \sqrt{\sigma_-(s)} \right)^n \text{H}_n \left( i \sqrt{\frac{T}{1-T}} \frac{\hat{x}_\theta}{\sigma_-(s)} \right), \quad (29)$$

$$\hat{X}_{2\mathcal{F}}^{(n)} = \left( \frac{\sqrt{T}}{2i} \sqrt{\sigma_+(s)} \right)^n \text{H}_n \left( i \sqrt{\frac{1-T}{T}} \frac{\hat{x}_{\theta+\pi/2}}{\sigma_+(s)} \right), \quad (30)$$

where  $\text{H}_n(z)$  is the  $n$ -th Hermite polynomial and we have scaled the results by setting  $\alpha = \sqrt{2}$ . For the purpose of further discussion we write explicitly the first two operational moments for both quadratures

$$\hat{X}_{1\mathcal{F}}^{(1)} = \sqrt{T} \frac{\hat{x}_\theta}{\sqrt{\sigma_-(s)}}, \quad (31)$$

$$\hat{X}_{1\mathcal{F}}^{(2)} = T \frac{\hat{x}_\theta^2}{\sigma_-(s)} + \frac{1-T}{2} \sigma_-(s), \quad (32)$$

$$\hat{X}_{2\mathcal{F}}^{(1)} = \sqrt{1-T} \frac{\hat{x}_{\theta+\pi/2}}{\sqrt{\sigma_+(s)}}, \quad (33)$$

$$\hat{X}_{2\mathcal{F}}^{(2)} = (1-T) \frac{\hat{x}_{\theta+\pi/2}^2}{\sigma_+(s)} + \frac{T}{2} \sigma_+(s). \quad (34)$$

We see that for  $T = 1$  or  $T = 0$  we are restricted to the measurement of either  $\hat{X}_{1\mathcal{F}}^{(n)}$  or  $\hat{X}_{2\mathcal{F}}^{(n)}$ , which are, in this case, equal to the powers of the intrinsic operators.

For simplicity, let assume that the phase of the squeezing vacuum is such that  $2\theta + \phi = 0$ , then the functions  $\sigma_{\pm}(s)$  becomes simpler

$$\sigma_{\pm}(s) = e^{\pm 2s}, \quad (35)$$

and the operational spread defined as  $\delta\hat{X}_{i\mathcal{F}} \equiv \sqrt{\langle\hat{X}_{i\mathcal{F}}^{(2)}\rangle - \langle\hat{X}_{i\mathcal{F}}^{(1)}\rangle^2}$  reads for both quadratures

$$(\delta\hat{X}_{1\mathcal{F}})^2 = Te^{2s}(\Delta x_{\theta})^2 + (1 - T) \frac{e^{-2s}}{2}, \quad (36)$$

$$(\delta\hat{X}_{2\mathcal{F}})^2 = (1 - T)e^{-2s}(\Delta x_{\theta+\pi/2})^2 + T \frac{e^{2s}}{2}, \quad (37)$$

where  $\Delta x_{\theta}$  is the intrinsic quantum dispersion.

We note that for the selected filter parameters one of the operational quadratures is squeezed and the second one is enhanced. This behavior is the same as for the intrinsic quadratures [18], however the measuring device introduces an additional factor contributing to the whole operational spread. Each operational spread contains two parts, the first part is the intrinsic uncertainty weighted by the factor  $T$  (or  $1 - T$ ) and the second part is the noise with the weight  $1 - T$  (or  $T$ ) introduced by the experimental device. The origin of these additional terms is clear. The NFM experimental setup enables us to perform a *joint* measurement of two quadratures in the extended Hilbert space. In the reduced space of the signal mode we pick up an additional noise due to the leaking noise mode through the unused ports in the NFM apparatus.

By changing the transmittance coefficient we can select which quadrature will be measured and how precise this measurement will be. If we set  $T = 0$  or  $T = 1$  we measure only one quadrature and the moments of the operational operators equal to the powers of the intrinsic homodyne observables. The operational result reduces to the intrinsic observable in this case because we have assumed perfect photodetectors with 100% efficiency.

The non ideal detectors can be described by a homodyne setup with perfect detectors, and a beam splitter with a certain effective transmittivity. The beam splitter mixes the incoming signal with the vacuum field and the results of the calculations are formally the same as those carried on for the non ideal detector [14,20].

It's worth to notice that in fact the NFM experimental scheme may be used for such a description of a quadrature measurement with a non ideal photodetector. In such a case we may forget about the upper right part of the NFM apparatus and see that the setup reduces to the situation described in [20]. In such a scheme there is a beam splitter in front of a pair of ideal detectors. If we put the squeezing parameter equal to 0, the operational observables obtained in this paper reduce to those obtained in [14].

We see that there are two possible and mathematically equivalent descriptions of the operational measurement performed with help of the NFM device. In one description the experiment is a simultaneous measurement of two operational quadratures with the help of ideal photodetectors with the NFM apparatus noise. In the other description, we perform a standard homodyne experiment, with a non unit quantum efficiency of the photo-detectors simulated by the beam splitter. Such a beam splitter is mixing the probe signal with the



vacuum state and the effective transmittance coefficient plays the role of the photodetector efficiency.

Another possible choice of the parameters leads us to the operational momentum and position operators derived in [8], indeed for  $T = \frac{1}{2}$ ,  $s = 0$  and  $|\alpha| = 2$ ,  $\theta = 0$  our results can be reduced to the special case of the results from this reference, with the filter being in the vacuum state.

The of operational operators  $\hat{X}_{1\mathcal{F}}^{(n)}$  become then the operational moments of the position operator whereas  $\hat{X}_{2\mathcal{F}}^{(n)}$  the operational moments of the momentum operator.

## V. SQUEEZED QUANTUM TRIGONOMETRY

Originally the NFM experiments have been devoted to the measurement of the relative phase between a high-intensity classical field and a low-intensity laser field. The quantities which were investigated in these experiments were built out of the field quadratures (19) and (20) ( with transmittance  $T = \frac{1}{2}$ ). These operators have been constructed in such a way that they can play the role of the commuting “sine” and “cosine” operators. As a result the phase of the quantum probe field has been defined operationally, avoiding all the problems associated with the construction of a Hermitian phase operator in quantum mechanics [21]. It was shown [11] that one can associate with the NFM experiment an operational phase operator.

In this section we extend this theoretical analysis for the case of the noise field being in a squeezed vacuum state (the original NFM experiments have been done with the vacuum state). We will show how to define an operator corresponding to any function of the phase. As a example we derive operators corresponding to the trigonometric functions. These operators form the squeezed quantum trigonometry of operational observables for the NMF setup.

We shall start our discussion, introducing an amplitude marginal of the POVM for the NFM device

$$\hat{\mathcal{F}}(\varphi) = \int dI \hat{\mathcal{F}}(\alpha = \sqrt{I}e^{i\varphi}). \quad (38)$$

From this POVM, we construct the corresponding phase propensity

$$\text{Pr}(\varphi) = \langle \hat{\mathcal{F}}(\varphi) \rangle, \quad (39)$$

which is a periodic function of phase, normalized to unity

$$\int \frac{d\varphi}{2\pi} \text{Pr}(\varphi) = 1. \quad (40)$$

### A. Phasor Basis

We define here the set of operational operators, which in the classical limit can be reduced to the Fourier expansion basis. The problem of constructing the operators corresponding to

the classical phase dependent quantities has been solved in Ref. [22]. Because every periodic function can be expressed in terms of its Fourier decomposition

$$f(\varphi) = \sum_{k=-\infty}^{\infty} f_k e^{ik\varphi}, \quad (41)$$

we will construct the operational quantum analogs of  $e^{ik\varphi}$  — so called *phasors* — in the following way

$$\overline{e^{ik\varphi}} \equiv \langle \hat{E}^{(k)} \rangle, \quad k = \pm 1, \pm 2, \dots \quad (42)$$

The reality of the propensity implies

$$\hat{E}^{(-k)} = \hat{E}^{(k)\dagger}, \quad (43)$$

because of the normalization condition (40) we have  $\hat{E}^{(0)} = 1$ .

As we have seen in the previous sections, the function  $f(\varphi)$  corresponds in the NFM device to the following operational operator

$$\hat{F} = \sum_{k=-\infty}^{\infty} f_k \hat{E}^{(k)}, \quad (44)$$

which is a function of the boson creation and annihilation operators of the probe field.

Because we have derived the POVM operator  $\hat{\mathcal{F}}$  for the NFM setup with an arbitrary noise field (11), we can derive the explicit expression for the phasor  $\hat{E}^{(k)}$  straight from the definition (6). Once these operators are known, the operational quantum operator can be calculated using a Fourier series expansion. Because of this we will call the set  $\{\hat{E}^{(k)}, k = 0, 1, 2, \dots\}$  the phasor basis. The definition of the phasors introduced in [22] requires, that these operators become in the classical limit functions of the phase only.

In order to find the phasor basis we start with the formula obtained in [11]

$$E^{(k)} = \left( \frac{\alpha}{\alpha^*} \right)^{k/2} \text{Tr}_v \left\{ U^k \hat{S}(s, \phi) |0_v\rangle \langle 0_v| \hat{S}^\dagger(s, \phi) \right\}, \quad (45)$$

where we have replaced the vacuum state of the filter by the squeezed vacuum state.

The complex number  $\alpha$  denotes the amplitude of the classical reference field of the local oscillator. The unitary operator  $\hat{U}$  is defined as follows

$$\hat{U} = \frac{\hat{X}_1 + i\hat{X}_2}{\sqrt{\hat{X}_1^2 + \hat{X}_2^2}} = \frac{\sqrt{\hat{X}_1^2 + \hat{X}_2^2}}{\hat{X}_1 - i\hat{X}_2}, \quad (46)$$

where  $\hat{X}_i$  are given by (19) and (20) with the beam splitter transmittivity  $T = \frac{1}{2}$ . Simple algebra gives

$$\hat{U} = \sqrt{\alpha^*/\alpha} \frac{\hat{b} + \hat{v}^\dagger}{\sqrt{(\hat{b} + \hat{v}^\dagger)(\hat{b}^\dagger + \hat{v})}} = \sqrt{\alpha^*/\alpha} \left( \frac{\hat{b} + \hat{v}^\dagger}{\hat{b}^\dagger + \hat{v}} \right)^{1/2}. \quad (47)$$

We recognize in this expression the unimodular operator (8). In this way we obtain an equivalent expression for  $\hat{E}^{(k)}$

$$\hat{E}^{(k)}(s, \phi) = \text{Tr}_v \left\{ \frac{(\hat{b} + \hat{v}^\dagger)^k}{[(\hat{b} + \hat{v}^\dagger)(\hat{b}^\dagger + \hat{v})]^{\frac{k}{2}}} \hat{S}(s, \phi) |0\rangle \langle 0| \hat{S}^\dagger(s, \phi) \right\}, \quad (48)$$

with  $k = 1, 2, 3, \dots$ . Negative values of  $k$  in (48) may be derived using (43).

Because operator  $\hat{U}$  obeys the assumptions imposed on the operator  $\hat{A}(\hat{b} + \hat{v}^\dagger, \hat{b}^\dagger + \hat{v})$  defined in (7) we can write the formula for the phasor using (10) as follows:

$$\hat{E}^{(k)}(s, \phi) = \int \frac{d^2\alpha}{\pi} \frac{\alpha^k}{(\alpha^*\alpha)^{\frac{k}{2}}} |\alpha, s\rangle \langle \alpha, s|, \quad (49)$$

obviously  $\hat{\mathcal{F}}(s, \phi) = |\alpha, s\rangle \langle \alpha, s|$  is the POVM associated with the described experimental scheme. It is clear, that the POVM, contrary to spectral measure, depends on experimental device. In fact, for each value of squeezing parameter  $s$  we obtain a different propensity a different POVM and a different phasor basis, even though the probe field remains unchanged.

An exact formula for the phasor may be derived straight from (49),

$$\hat{E}^{(k)}(s, \phi) = \hat{S}(s, -\phi) : \left[ \frac{(\hat{b} \cosh s - \hat{b}^\dagger e^{i\phi} \sinh s)^k}{\hat{b}^\dagger \cosh s - \hat{b} e^{-i\phi} \sinh s} \right]^{\frac{k}{2}} : \hat{S}^\dagger(s, -\phi). \quad (50)$$

where  $:$  denotes the antinormal ordering. When the squeezing parameter  $s$  tends to zero the above formula reduces to the one from [11].

## B. Properties of the Squeezed Quantum Trigonometry

In this section we find various phase space representations for the simple combinations of the phasor basis. Using these relations we derive the trigonometric operational operators.

These trigonometric operators will be defined in accordance with (44). We have the following operators

$$\begin{aligned} \hat{S}^{(1)} &\equiv \frac{1}{2i}(\hat{E}^{(1)} - \hat{E}^{(-1)}), & \hat{S}^{(2)} &\equiv \frac{1}{2} - \frac{1}{4}(\hat{E}^{(2)} + \hat{E}^{(-2)}), \\ \hat{C}^{(1)} &\equiv \frac{1}{2}(\hat{E}^{(1)} + \hat{E}^{(-1)}), & \hat{C}^{(2)} &\equiv \frac{1}{2} + \frac{1}{4}(\hat{E}^{(2)} + \hat{E}^{(-2)}), \end{aligned} \quad (51)$$

that correspond to the trigonometric functions  $\sin \varphi$ ,  $\cos \varphi$ ,  $\sin^2 \varphi$  and  $\cos^2 \varphi$  respectively. As it was stressed earlier and follows from definition of the phasors  $(\hat{C}^{(1)})^2 \neq \hat{C}^{(2)}$ . Consequently, the Pythagorean theorem  $\sin^2 \varphi + \cos^2 \varphi = 1$  is obeyed by  $\hat{S}^{(2)}$ ,  $\hat{C}^{(2)}$ , but not by  $(\hat{S}^{(1)})^2$  and  $(\hat{C}^{(1)})^2$ .

In order to investigate the properties of the phasors we evaluate their phase space representations corresponding to a certain arbitrary  $\eta$ -ordering of boson operators, such a phase representation may be defined as follows [19]

$$E_{P^{(\eta)}}^{(k)}(s, \phi) = \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \alpha^*\beta} e^{\frac{\eta}{2}|\beta|^2} \text{Tr} \left\{ \hat{E}^{(k)}(s, \phi) \hat{D}(\beta) \right\}. \quad (52)$$

The number  $\eta$  lies between minus infinity and plus one and  $P^{(\eta)}$  denotes the corresponding quasidistribution. Especially interesting cases are  $P^{(-1)}$ ,  $P^{(0)}$ ,  $P^{(-1)}$ , which correspond to the  $Q$ -representation (normal ordering), the Wigner function (symmetric ordering) and the  $P$ -representation (antinormal ordering), respectively. In this formula the phase space integration can be done and as a result we obtain

$$E_{P^{(\eta)}}^{(k)}(s, \phi) = \int_{-\infty}^{\infty} \frac{d^k\lambda}{\sqrt{\pi^k}} \Omega^{\frac{k+1}{2}} H_k \left( \frac{1}{2} \sqrt{\Omega} \left[ \Sigma\beta + \lambda^2\beta^* e^{-i\phi} \sinh 2s \right] \right) \times \exp \left\{ -\lambda^2 \Omega \left[ \Sigma|\beta|^2 + \frac{1}{4} \lambda^2 \sinh 2s \left( e^{i\phi} \beta^2 + e^{-i\phi} \beta^{*2} \right) \right] \right\}, \quad (53)$$

where

$$\Omega \equiv \left[ \left( e^{-s} \left( 1 - \lambda^2 \frac{\eta+1}{2} \right) + \lambda^2 \cosh s \right) \left( e^s \left( 1 - \lambda^2 \frac{\eta+1}{2} \right) + \lambda^2 \cosh s \right) \right]^{-1} \quad (54)$$

and

$$\Sigma \equiv \lambda^2 \left( \cosh^2 s - \frac{\eta+1}{2} \right) + 1. \quad (55)$$

This result is meaningful only if  $\Omega$  is real, otherwise one of the integrals, that appeared earlier, is divergent. Because the most interesting cases are the  $Q$ ,  $P$ -representations and the Wigner function we have

$$\Omega = \begin{cases} \left[ (\lambda^4 + 2\lambda^2) \cosh^2 s + 1 \right]^{-1} & \text{for } \eta = -1 \\ \left[ \frac{1}{4} \lambda^4 + \lambda^2 (2 \sinh^2 s + 1) + 1 \right]^{-1} & \text{for } \eta = 0 \\ \left[ 1 - (\lambda^4 + 2\lambda^2) \sinh^2 s \right]^{-1} & \text{for } \eta = 1. \end{cases} \quad (56)$$

For  $\eta = 1$ ,  $\Omega$  becomes a complex number, so the  $P$ -representation for the squeezed phasors does not exist unless we tend with  $s$  to zero. In this limit we retrieve the original NFM phasors from [11]. In our further analysis we shall restrict the calculations to the Wigner function only, but all the results may be easily rewritten for the respective  $Q$ -representation.

If we write  $\beta = \sqrt{I} e^{i\theta}$  and use (53), the definitions (51) give for  $\eta = 0$  the “sine” and the “cosine” Wigner functions

$$C_W^{(1)}(s, \phi) = \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\pi}} \exp \left\{ \frac{-\lambda^2 I [1 - \frac{1}{2} \lambda^2 + \lambda^2 \cosh^2 s (1 + \tanh s \cos(\phi + 2\theta))]}{\frac{1}{4} \lambda^4 + \lambda^2 (1 + 2 \sinh^2 s) + 1} \right\} \times \frac{\sqrt{I} [\cos \theta - \frac{1}{2} \lambda^2 \cos \theta + \lambda^2 \cosh^2 s (\cos \theta + \tanh s \cos(\phi + \theta))]}{[\frac{1}{4} \lambda^4 + \lambda^2 (1 + 2 \sinh^2 s) + 1]^{\frac{3}{2}}} \quad (57)$$

and

$$S_W^{(1)}(s, \phi) = \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\pi}} \exp \left\{ \frac{-\lambda^2 I [1 - \frac{1}{2} \lambda^2 + \lambda^2 \cosh^2 s (1 + \tanh s \cos(\phi + 2\theta))]}{\frac{1}{4} \lambda^4 + \lambda^2 (1 + 2 \sinh^2 s) + 1} \right\} \times \frac{\sqrt{I} [\sin \theta - \frac{1}{2} \lambda^2 \sin \theta + \lambda^2 \cosh^2 s (\sin \theta - \tanh s \sin(\phi + \theta))]}{[\frac{1}{4} \lambda^4 + \lambda^2 (1 + 2 \sinh^2 s) + 1]^{\frac{3}{2}}}. \quad (58)$$

Similar expressions for the Wigner functions  $S_W^{(2)}(s, \phi)$  and  $C_W^{(2)}(s, \phi)$  can be derived from the phasor basis. Due to the lengthy mathematical form, we shall present in this paper only plots of these functions. The Wigner functions of the operational quantum trigonometry provide a simple framework for the physical discussion of the obtained results. In Fig. 2, we have plotted the Wigner function  $S_W^{(1)}(2, \frac{\pi}{2})$ . For field intensities tending to infinity we should reproduce the classical trigonometry. In order to see this we recall that  $\lim_{I \rightarrow \infty} \left( \sqrt{\frac{I}{\pi}} \right) e^{-I f^2(z)} = \delta(f(z))$ . After some simple algebra we see that  $f(\lambda)$  in (57) and (58) has only one zero point at  $\lambda_0 = 0$  and the modulus of a derivative of  $f(\lambda)$  in that point is one. As a result, using the well known relation between  $\delta(f(z))$  and  $\delta(z)$ , we see that in order to perform the classical limit, we have to put only  $\lambda = 0$  in the integrands of the Wigner functions. In this limit

$$\begin{aligned} \lim_{I \rightarrow \infty} S_{Q \text{ or } W}^{(1)}(s, \phi) &= \sin \theta \\ \lim_{I \rightarrow \infty} C_{Q \text{ or } W}^{(1)}(s, \phi) &= \cos \theta. \end{aligned} \quad (59)$$

This result is also seen on Fig 2. where indeed for large values  $I$  we reconstruct the classical trigonometric function.

As it might have been expected, in the classical limit the dependence on the squeezing parameters disappears, in agreement with the fact that squeezing is a purely quantum effect.

Similarly we can prove, that  $\hat{S}^{(2)}(s, \phi)$  and  $\hat{C}^{(2)}(s, \phi)$  possess a proper classical limit. For these functions we present only graphs. As it can be seen on Fig. 3 and Fig. 4, the classical limit of the Wigner function of  $\hat{C}^{(2)}(s, \phi)$  reproduces the  $\cos^2 \varphi$ .

We shall discuss now the limit of a very small probe field intensity. Let assume for a while, that the phase of squeezing parameter  $\phi$  is zero, and  $I \rightarrow 0$ . In this case we can expand the exponential factor of the integrand in (58) and (57). Keeping only the first term we obtain

$$\begin{aligned} C_W^{(1)}(s, \phi) &\stackrel{I \rightarrow 0}{\approx} \sqrt{I} \mathcal{A}_+(s, \phi = 0) \cos \theta \\ S_W^{(1)}(s, \phi) &\stackrel{I \rightarrow 0}{\approx} \sqrt{I} \mathcal{A}_-(s, \phi = 0) \sin \theta \end{aligned} \quad (60)$$

where

$$\mathcal{A}_{\pm}(s, \phi = 0) = \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\pi}} \frac{1 + (1/2)\lambda^2 e^{\pm 2s}}{[\frac{1}{4}\lambda^4 + \lambda^2(1 + 2 \sinh^2 s) + 1]^{\frac{3}{2}}} \quad (61)$$

are two different amplitudes for the “sine” and the “cosine” Wigner functions. These amplitudes depend only on the squeezing parameter. This result shows the effect of the amplitude squeezing near the vacuum. From (61) we see that for  $\phi = 0$   $S_W^{(1)}(s, 0)$  is strongly squeezed then  $C_W^{(1)}(s, 0)$  (i.e. the amplitude of sine is smaller than that for cosine). Obviously for  $I = 0$  the amplitudes are exactly equal to zero, which is in agreement with the common intuition that the phase of a light field in the vacuum state is randomly distributed.

Changing the squeezing phase to  $\phi = \pi$ , we reverse the amplitude squeezing from the sine to the cosine function. It is simple to verify, that the amplitudes for different values of  $\phi$  are connected in the following way  $\mathcal{A}_+(s, \phi = 0) = \mathcal{A}_-(s, \phi = \pi)$  and  $\mathcal{A}_-(s, \phi = 0) = \mathcal{A}_+(s, \phi = \pi)$ , we see that in this case, that  $C_W^{(1)}(s, 0)$  is strongly squeezed then  $S_W^{(1)}(s, 0)$ .

For arbitrary values of  $\phi$  and  $s$  the separation of the cosine Wigner function into an amplitude, and a purely angle dependent part is no longer possible, but a clear squeezing of the amplitude can also be observed comparing, for example, Fig. 3 ( $s=0.5$ ) and Fig. 4 ( $s=1.5$ ). For  $s = 2$  (Fig. 2) the operational phasor is by far more squeezed than the two functions from Fig. 3 and Fig. 4. Note that the range of the intensity  $I$  is almost twice bigger than the range for Fig. 3 and 4.

Similarly we can find an asymptotic expression for  $C_W^{(2)}(s, \phi)$ . In the limit of small  $I$

$$C_W^{(2)}(s, \phi) = \frac{1}{2}(1 - c(s, \phi)), \quad (62)$$

where  $c(s, \phi)$  is an  $I$ -independent function of the squeezing parameter

$$c(s, \phi) = \int_{-\infty}^{\infty} \frac{d\lambda_1 d\lambda_2}{\pi} \frac{(1/2)(1 + (1/4)\lambda^4 + \lambda^2(1 + 2\sinh^2 s)) \sinh 2s \cos \phi}{[\frac{1}{4}\lambda^4 + \lambda^2(1 + 2\sinh^2 s) + 1]^{\frac{5}{2}}}, \quad (63)$$

where  $\lambda$  is understood here as a norm of the two component vector  $\vec{\lambda} = [\lambda_1, \lambda_2]$ . It's easy to check, that  $c(s, \phi)$  tends either to unity ( $\phi = 0$ ) or to minus unity ( $\phi = \pi$ ). As a result, for small intensities  $C_W^{(2)}(s, \phi)$  becomes zero ( $\phi = 0$ ) or one ( $\phi = \pi$ ). For  $\phi = \pi/2, 3\pi/2$  we have  $c(s, \phi) = 0$  and  $\lim_{I \rightarrow 0} C_W^{(2)}(s, \phi) = 1/2$ .

For small  $I$  the squeezing influences the system very strongly. If the squeezed phase  $\phi$  equals to  $\pi/2$ , the  $\hat{C}^{(2)}$  Wigner function tends to  $1/2$  (Fig. 3), whereas for  $\phi = 0$  it takes values near zero (Fig. 4). Such a dramatic change of the cosine quadratures occurs because in the limit of small  $I$ , purely quantum effects of the squeezed vacuum are important. The squeezing allows one of the quadratures to be reduced below the vacuum level represented by a uniformly distributed random phase. The uniform distribution of the phase corresponding to the vacuum state leads to an operational Wigner function for  $I = 0$  equal to  $\frac{1}{2}$ . For a squeezed vacuum, this uniform random-phase distribution is modified [23] and a significant change of the operational quadrature is possible. In fact fluctuations below  $\frac{1}{2}$  in the  $\hat{C}^{(2)}$  Wigner function exhibit the quantum nature of the squeezed vacuum. Those results are even more clear in view of our previous analysis of operational quadratures (19) and (20). We see straight from the definition of  $\hat{U}$  (46), that operational operators  $\hat{C}^{(2)}(s, \phi)$  and  $\hat{S}^{(2)}(s, \phi)$  are just normalized homodyne quadratures, so their dependence on squeezing phase and  $s$  may be understood as if they were homodyne quadratures. These operators are also quantum analogs of certain classical functions of the phase of the electromagnetic field. Because of this they possess a much richer structure and this enables, for example, to investigate the phase properties of the quantum optical fields.

## VI. CONCLUSIONS

We have presented an operational theory of the eight-port homodyne detection scheme. For such a measuring apparatus, we have derived the operational quadrature operators for an arbitrary noise field leaking through the unused port. In the framework of the operational approach we have derived the quantum propensity, the POVM and the corresponding operational observables for the NFM device. For the noise field being the squeezed vacuum we have constructed the quantum trigonometry and the corresponding Wigner functions.

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# FIGURES

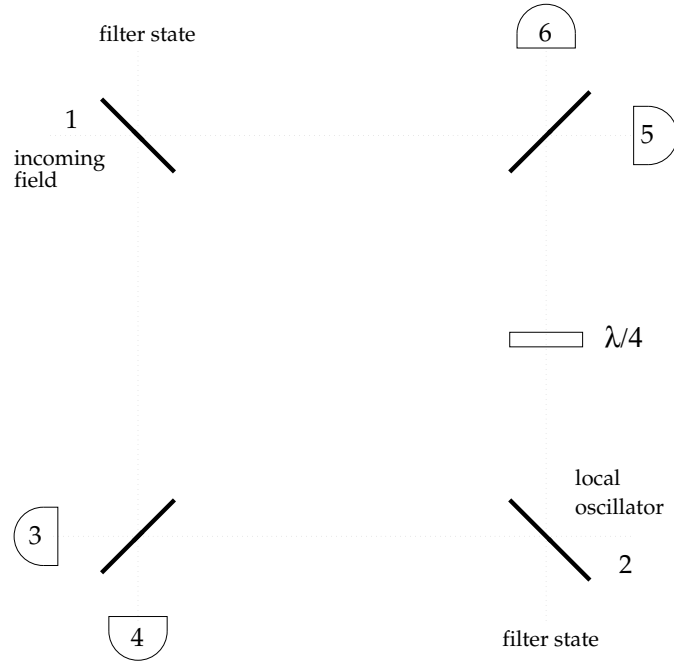


FIG. 1. The NFM apparatus. The photodetectors are labeled by numbers 3, 4 and 5, 6, whereas the beam splitters are represented by dark thickened lines

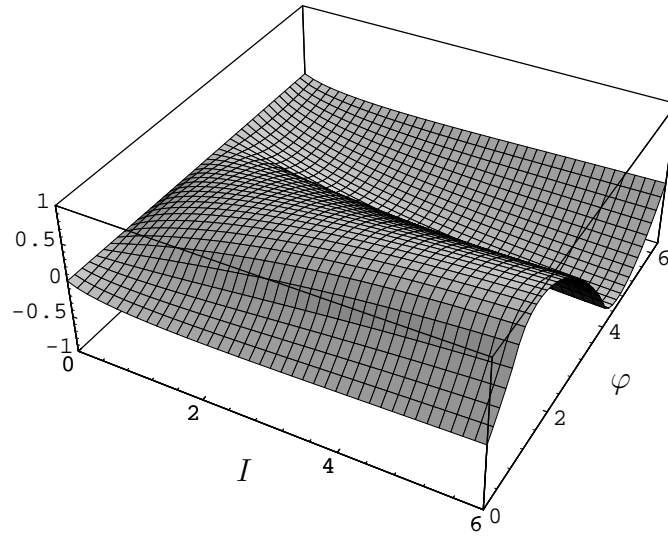


FIG. 2. Plot of the Wigner function of the operational operator  $\hat{S}^{(1)}(s=2, \phi=\pi/2)$ .

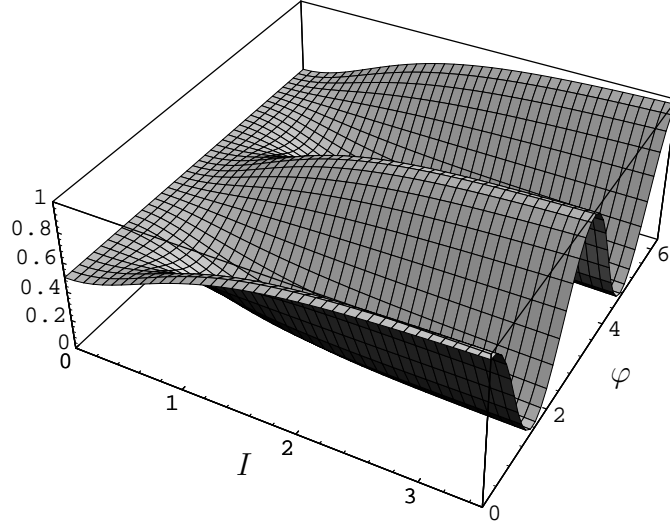


FIG. 3. Plot of the Wigner function of the operational operator  $\hat{C}^{(2)}(s = 0.5, \phi = \pi/2)$ .

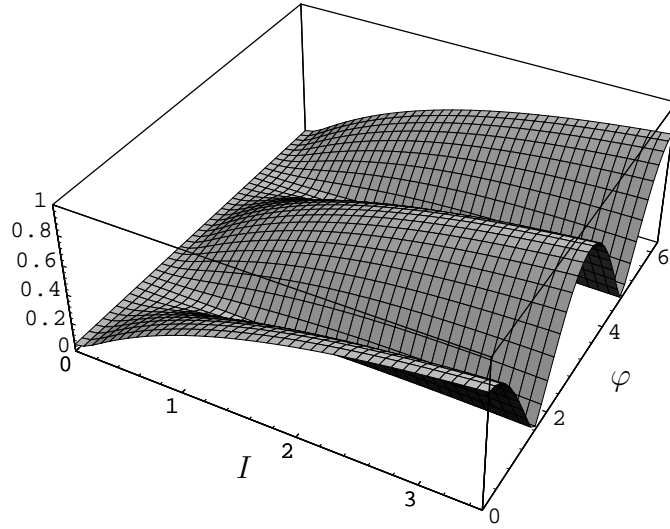


FIG. 4. Plot of the Wigner function of the operational operator  $\hat{C}^{(2)}(s = 1.5, \phi = 0)$ .